

Course review.

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Let $\Omega \subset \mathbb{C}$ be a simply-connected region

The following are equivalent definitions of an analytic function in Ω :

1) $\forall z \in \Omega : \exists \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z} = f'(z)$.

2) f is real-differentiable in Ω

and $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = 0$ (Cauchy-Riemann)

in real form: $f = u + iv : \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

3) f is real-differentiable in Ω and $\forall z$:

$f(z+h) = f(z) + T_z(h) + o(h)$, where T_z is complex-linear.

4) $\forall \gamma \subset \Omega$ - closed, $\oint_{\gamma} f(z) dz = 0$. (Cauchy Theorem)

$\gamma \sim 0$
in general.

4sc) $\exists F$ - analytic in Ω , $f = F'$.

5) $\forall z \in \Omega \exists r > 0 : \forall R \subset B(z, r)$ - rectangle with sides parallel to the axes, $\oint_{\partial R} f(z) dz = 0$.

6) $\forall \gamma$ - closed: $f(z) n(\gamma, z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$. (Cauchy integral formula)

$\gamma \sim 0$
in general

7) $f \in C^\infty(\Omega) \wedge$ Cauchy-Riemann

8) $\forall z_0 \in \Omega, \exists r > 0 : |z - z_0| < r \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.

9) $\forall z_0 \in \Omega \quad |z - z_0| < \text{dist}(z_0, \partial\Omega) \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.

$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$. where

$r < \text{dist}(z_0, \partial\Omega), \quad C_r = \{z_0 + re^{it}, 0 \leq t \leq 2\pi\}$.

If Ω is an arbitrary region then

4sc) does not hold, but 1)-3), 5), 7)-9) still hold, 4), 6) hold for $\gamma \sim 0$.

For 4sc) - take any $\gamma \neq 0$ in $\Omega, z_0 \notin \Omega : n(\gamma, z_0) \neq 0$.

Then $\frac{1}{z - z_0}$ does not have antiderivative.

General Cauchy Integral Formula:

Let γ be a cycle, $\gamma \sim 0$ in Ω . $f \in A(\Omega)$.

Then if $z \notin \gamma, z \in \Omega$, then

$n(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\gamma} f(\zeta) d\zeta$

When $\gamma \ni z \neq \gamma, z \in \mathbb{C}$, then

$$h(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

General Argument Principle.

If $\gamma \sim 0$ in \mathbb{C} , $f \in \mathcal{M}(\mathbb{C})$, $f \neq 0$ on γ , then

$$h(\gamma, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum_{z \in \mathbb{C}} h(\gamma, z) \text{ord}(f, z).$$

Complex numbers:

$$z = x + iy = |z|e^{i \arg z}, \arg z = \{ \theta, \theta + 2\pi k, k \in \mathbb{Z} \}$$

Multiplication by z : rotation by $\arg z$, dilation by $|z|$.

$$\hat{\mathbb{C}}: \mathbb{C} \cup \{\infty\}, \quad d(z, z') = \frac{2|z - z'|}{\sqrt{1+|z|^2} \sqrt{1+|z'|^2}}, \quad d(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$$

$$e^z = e^x \cdot e^{iy} = e^x (\cos y + i \sin y) \quad |e^z| = e^x, \arg e^z = \{y + 2\pi k, k \in \mathbb{Z}\}$$

$$\boxed{e^{z+i2\pi} = e^z}$$

$$z \neq 0 \quad \log z = \log |z| + i \arg z$$

$\text{Log } z = \log |z| + i \text{Arg } z$ not continuous on $\mathbb{R}_- = (-\infty, 0]$!

Example. $\text{Log}(e^{3+5i}) \neq 3+5i$
 $= 3 + (5 - 2\pi)i$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

Fractional Linear (Möbius) transformations.

$$f(z) = \frac{az+b}{cz+d} \in \mathcal{M}(\hat{\mathbb{C}}), \text{ rational of degree 1.}$$

Normalization: $a d - bc = 1$.

- Preserve:
- 1) lines and circles
 - 2) symmetries wrt lines and circles.
 - 3) Cross ratio.

Example: find all conformal maps from $\{\text{Re } z > 0\}$ onto \mathbb{D} .

Know: need to find one, the rest - using self-map of \mathbb{D} .

So if we find one Möbius bijection, the rest will be Möbius, as compositions of Möbius maps.

$\varphi: \{Re z > 0\} \rightarrow \mathbb{D}$ - Möbius.

Let $\varphi(z_0) = 0$, then $\varphi(-\bar{z}_0) = \infty$.

So $\varphi(z) = c \frac{z - z_0}{z + \bar{z}_0}$ for some c .

$\partial \mathbb{D} \ni \varphi(i) = c \frac{i - z_0}{i + \bar{z}_0} \Rightarrow |c| = 1$.

So $\varphi(z) = e^{i\theta} \frac{z - z_0}{z + \bar{z}_0}$, $Re z_0 > 0$ - general form.

If $z = it$ ($t \in \mathbb{R}$), $|\varphi(z)| = \left| \frac{it - z_0}{it + \bar{z}_0} \right| = \left| \frac{it - z_0}{-(it - z_0)} \right| = 1$.

So $\varphi: i\mathbb{R} \rightarrow \partial \mathbb{D}$.
 $\varphi(z_0) = 0$ - indeed maps $\{Re z > 0\}$ onto \mathbb{D} .

Locally uniform convergence:

Let (f_n) be a sequence of functions on a region Ω .

TEAE:

1) $\forall K \subset \Omega$ compact, $f_n \rightarrow f$ uniformly on K .

2) $\forall z \in \Omega \forall \varepsilon > 0 \exists \delta(\varepsilon, z), N(\varepsilon, z): \left\{ \begin{array}{l} |w - z| < \delta \\ w \in \Omega \\ n > N \end{array} \right\} \Rightarrow |f(w) - f_n(w)| < \varepsilon$.

Special cases:

- Meromorphic: $\forall K \exists N(K): n > N(K) \Rightarrow f_n \in \underline{A}(K)$.

- Products: $\forall K \exists N(K): n > N(K) \Rightarrow f_n \neq 0$ on K .

$f_n \in A(\Omega), f_n \xrightarrow{l.u.} f \Rightarrow f \in A(\Omega)$ - Weierstrass
 $f_n^{(k)} \rightarrow f^{(k)}$ ✓

(f_n) - locally uniformly bounded, $\exists z_0: \forall K: f_n^{(k)}(z_0) \rightarrow f^{(k)}(z_0) \Rightarrow f_n \xrightarrow{l.u.} f$ - Vitaly.

$f_n \neq 0 \Rightarrow \begin{array}{l} f = 0 \\ \text{or } f \neq 0 \end{array}$ - Hurwitz

f_n - conformal $\Rightarrow \begin{array}{l} f \text{ - conformal} \\ \text{or } f = \text{const.} \end{array}$ - Hurwitz.

Maximum principle:

$f \in A(\Omega) \Rightarrow$ no $z_0 \in \Omega$ is a local maximum of $|f(z)|$
if $f(z_0) \neq 0$, - not a local minimum (consider $\frac{1}{f(z)}$).

Uniqueness Theorem: $f, g \in \mathcal{A}(\Omega)$,

$$A = \{z : f(z) = g(z)\}$$

A has a limit point in $\Omega \Rightarrow f \equiv g$.

Harmonic functions.

Let $u \in C(\Omega)$, real-valued.

$u \in \text{Harm}(\Omega)$ if one of the following holds.

1) $u \in C^2(\Omega)$, $\Delta u \equiv 0$.

2) $\forall z \in \Omega \exists r_0 \leq \text{dist}(z, \partial\Omega)$:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt, \quad 0 < r < r_0.$$

3) $\forall z \in \Omega \forall r < \text{dist}(z, \partial\Omega)$

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt.$$

4) $\forall z \in \Omega \forall r < \text{dist}(z, \partial\Omega), \forall w \in B(z, r)$:

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |w|^2}{|z + re^{it} - w|^2} u(z + re^{it}) dt$$

5) $\forall z \in \Omega \ r_0 = \text{dist}(z, \partial\Omega)$

$$\exists f \in \mathcal{A}(B(z, r_0)), \quad u = \text{Re} f.$$

5_{sc}) If Ω is simply-connected,

$$\exists f \in \mathcal{A}(\Omega): \quad u = \text{Re} f.$$